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Normalized solutions of Kirchhoff equations with Hartree-type nonlinearity

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Abstract

In the present paper, we prove the existence of the solutions $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ to the following Kirchhoff equations with the Hartree-type nonlinearity under the general mass supercritical settings,

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u - \lambda u = \left[I_\alpha * (K(x)F(u))\right] K(x) f(u), \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $a, b > 0$ are prescribed, $I_\alpha = |x|^{\alpha-3}$ is the riesz potential where $\alpha \in (0,3)$, $K \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^+)$ satisfies an explicit assumption and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies some weak conditions, we develop some new tricks for dealing with the Hartree-type term to overcome the difficulties produced by the appearance of non-constant potential $K(x)$. This paper extends and promotes the previous results on prescribed L^2-norm solutions of the Kirchhoff-type equation.

Key Words: Kirchhoff problem, Normalized solution, Hartree-type nonlinearity, mass supercritical case.
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1 Introduction

In this paper, we are concerned with the following Kirchhoff equation with the Hartree-type nonlinearity:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u-\lambda u=\left(\int_{\mathbb{R}^3}\frac{K(y)F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}y\right)K(x)f(u(x)), \\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a,b>0$ are prescribed, $\alpha\in(0,3)$, λ is unknown and will appear as a Lagrange multiplier, $K\in\mathcal{C}^1(\mathbb{R}^3,\mathbb{R}^+)$ and $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$ satisfies

(F1) $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$, there exists $\mu\in\left(\frac{7+\alpha}{3},3+\alpha\right)$ such that $0\le f(t)t\le\mu F(t)$ for all $t\in\mathbb{R}$, and $\mathrm{meas}\{t\in\mathbb{R}:\mu F(t)-f(t)t=0\}=0$;

(F2) there exists $\theta\in\left(1+\frac{\alpha}{3},\frac{7+\alpha}{3}\right)$ such that

$$\lim_{|t|\to0}\frac{F(t)}{|t|^\theta}=0\quad\text{and}\quad\lim_{|t|\to\infty}\frac{F(t)}{|t|^{\frac{7+\alpha}{3}}}=+\infty;$$

(F3) the mapping $t\mapsto[f(t)t-\theta F(t)]/|t|^{\frac{4+\alpha}{3}}t$ is nondecreasing on $(-\infty,0)$ and $(0,+\infty)$.

Such kind of equation (1.1) involving general nonlinearity $f(u)$, named as Kirchhoff-type equation, was first proposed by Kirchhoff [14] as an extension of the classical D'Alembert's wave equations for free vibration of elastic strings. For more details about the physical background on Kirchhoff's model, we refer to [1, 2, 5].

From the mathematical point of view, the term $\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x$ in (1.1) indicates that the Kirchhoff-type equation is not a pointwise identity any more. It is worth mentioning that the pioneer work by Lions [17] first introduced a functional analysis approach, hereafter, a series of subsequent study has been done on the existence and multiple existence of solutions to the nonlinear Kirchhoff equations. When $\lambda\in\mathbb{R}$ is a fixed parameter, we call it fixed frequency problem, or λ replaces with general potential $V(x)$, the existence of solutions around the Kirchhoff equations has been intensively studied during the last decade, we refer to [3, 8, 9, 12, 15, 24] and the references therein. In this case, one can apply the variational method to look for the critical points of the associated energy functional, such as the methods explored in [31, 32, 33], but without any information on the L^2-norm of the solutions.

In the present paper, motivated by the fact that physicists are often interested in "normalized solutions", we search for solutions in $H^1(\mathbb{R}^3)$ possessing the prescribed L^2-norm. More precisely, for given $c > 0$,

$$(\mu_c, u_c) \in \mathbb{R} \times H^1(\mathbb{R}^3) \quad \text{solution of (1.1) with} \quad \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = c^2.$$

This kind of problem is naturally derived from the research of orbital stability of the standing waves for the time-dependent nonlinear Kirchhoff equation, and it seems to be particularly meaningful from the physical point of view as there is a conservation of mass. Such prescribed L^2-norm solutions of (1.1) can be obtained by looking for critical points of the following functional,

$$\begin{aligned} I(u) =& \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \right)^2 \\ & - \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(y)F(u(y))K(x)F(u(x))}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{1.2}$$

on the constraint

$$\mathcal{S}_c = \left\{ u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = c \right\},$$

where $F(u) = \int_0^u f(t)\mathrm{d}t$. In this case, the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, and each critical point $u_c \in \mathcal{S}_c$ of $I|_{\mathcal{S}_c}$ corresponds to a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that (u_c, λ_c) solves (weakly) problem (1.1).

On the one hand, among the investigations into the existence of normalized solutions of the Kirchhoff equation, Ye [26] first demonstrated the existence and non-existence of normalized solutions to the following equation

$$\begin{cases} -\left(a + b \displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 \right) \Delta u - \lambda u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.3}$$

for $p \in (2, 6)$. In particular, Ye [26] used the minimization method to find the minimizer of

$$\tilde{\sigma}(c) := \inf_{u \in \tilde{\mathcal{S}}_c} \tilde{I}(u), \tag{1.4}$$

where $\tilde{\mathcal{S}}_c = \left\{ u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c \right\}$ and $\tilde{I}(u)$ is the corresponding energy functional of the equation (1.3). Ye [26] succeeded to prove two existing results of $\tilde{\sigma}(c)$, the minimizer $\tilde{\sigma}(c)$ attained under different ranges of $p < 2 + 8/N$. Moreover, by a scaling technique and applying the concentration-compactness principle, they succeeded to verify that there is no minimizers for problem (1.3) when $p \geq 2 + 8/N$. In particular, for the case of $2 + 8/N < p < 2^*$, $\tilde{\sigma}(c) = -\infty$, which implies the minimization problem (1.4) is not available. In stead of performing the minimization argument on $\mathcal{S}_c$, Ye [26] introduced a suitable submanifold which is also a natural constraint of $I|_{\mathcal{S}_c}$. Under the help of which, Ye [26] could find the mountain pass critical point for the $\tilde{I}|_{\mathcal{S}_c}$. Based on the above fact, it is easy to know that the L^2-critical exponent $p = 2+8/N$ is the threshold exponent of problem (1.3), the corresponding functional is bounded below when $p < 2 + 8/N$ and unbounded below when $p \geq 2 + 8/N$. After the work by Ye [26], a serious of subsequent study has been done on the existence of normalized solutions to the nonlinear Kirchhoff equation, for L^2-critical problem we refer to [27, 28], where the author proved the existence of mountain pass critical point of (1.3) on $\tilde{\mathcal{S}}_c$ in [27] and investigated the asymptotic behavior of critical points in [28], then Luo & Wang [19] obtained the multiplicity existence of solutions with normalized L^2-norm of the equation (1.3) in $N = 3$ with $14/3 < p < 6$. When involving a trapping potential, Guo, Zhang & Zhou [10] obtained the existence and discussed the blow-up behavior of solutions with normalized L^2-norm. For more details about the normalized solutions of equation (1.1) with nonlinearities such as $|u|^{p-2}u$, we refer to [13, 20].

Concerning the case of general nonlinearities of the Kirchhoff equation, there seems to be only few relevant works [6, 11, 25, 30]. Xie & Chen [25] first generalized the previous work to the general nonlinearities f which satisfies $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{14/3}} = +\infty$. Recently, He *et al.* [11] proved the existence of ground state normalized solutions for any given $c > 0$, then they verified the asymptotic behavior of these solutions when $c \to 0^+$ as well as $c \to +\infty$. It is worth mentioning the very recent work by Zeng *et al.* [30], they used a global branch approach which does not depend on the geometry of the energy functional, so that they could handle the nonlinearities in a unified way, which are either mass subcritical, mass critical or mass supercritical.

On the other hand, we take a look into the results of normalized solutions involving the Hartree-type nonlinearity. Li & Luo [16] used a constrained minimization method on a suitable submanifold of $\tilde{\mathcal{S}}(c)$, then they proved the N-dimension fractional Choquard equation has a critical point on $\tilde{\mathcal{S}}(c)$ with the least energy among all the critical points of the corresponding functional restricted on $\tilde{\mathcal{S}}(c)$. Under the weaker conditions, Yuan, Chen & Tang [29] used a minimax procedure and some new analytical technique to show that for any $c > 0$, the 3-dimension Choquard equation possesses at least one normalized solution. Bartsch, Liu & Liu [4] used the stretched functional method to obtain a (PS) sequence for the corresponding functional on $\tilde{\mathcal{S}}(c)$ and together with concentration compactness argument to show the weak limit of this (PS) sequence is nontrivial, then the conclusion is easy to know after verifying the $\tilde{\sigma}(c)$ is strictly decreasing.

It seems that there is only one work [18] investigated the N-dimension Choquard equation involving kirchhoff type perturbation, which forms as (1.1) with $K(x) = 1$ and $f(u) = |u|^{p-2}u$, under different ranges of p, Liu [18] obtained the threshold values separating the existence and nonexistence of critical points, then Liu studied the behaviors of the Lagrange multipliers and the energies corresponding to the constrained critical points when $c \to 0$ and $c \to +\infty$ respectively. To the best of our knowledge, concerning the equation (1.1) involving the Hartree-type nonlinearity and term $K(x)$, there still exists a blank for the existence, which is our goal in this paper. The existing methods either reliy heavily on the power-type nonlinearity $f(u) = |u|^{p-2}u$ with $p \in (14/3, 6)$, or could not deal with the appearance of non-constant potential $K(x)$. Hence, the existing methods could not adapt directly to the equation (1.1).

In this paper, we will discuss the existence of normalized solutions for equation (1.1) with the non-constant potential function $K(x)$. To state our result, we make the following assumptions on K:

(K1) $K \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^+)$ and $0 < K_\infty := \lim_{|y|\to\infty} K(y) \le K(x)$ for all $x \in \mathbb{R}^3$;

(K2) $K \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^+)$ and $(3 + \alpha - \mu)K(x) + 2\nabla K(x) \cdot x \ge 0$ for all $x \in \mathbb{R}^3$, and $(\frac{3\theta-3-\alpha}{2})K(tx) - \nabla K(tx) \cdot tx$ is nonincreasing on $t \in (0, \infty)$ for every $x \in \mathbb{R}^3$.

By a standard argument we know that $I \in \mathcal{C}^1(H^1(\mathbb{R}^3), \mathbb{R})$. Before illustrating our result we introduce some notations,

$$\mathcal{M}_c := \left\{ u \in \mathcal{S}_c : J(u) := \frac{d}{dt} I(u^t)|_{t=1} = 0 \right\},$$

where

$$u^t(x) := t^{3/2} u(tx), \quad \forall\ t > 0, u \in H^1(\mathbb{R}^3),$$

it is easy to verify that $u^t \in \mathcal{S}_c$ for all $t > 0$ if $u \in \mathcal{S}_c$. Our main result is as follows.

Theorem 1.1. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then for any* $c > 0$*, problem* (1.1) *has a couple of solution* $(\bar{u}_c, \lambda_c) \in \mathcal{S}_c \times \mathbb{R}^-$ *such that*

$$I(\bar{u}_c) = \inf_{u\in\mathcal{M}_c} I(u) = \inf_{u\in\mathcal{S}_c} \max_{t>0} I(u^t) > 0.$$

Our paper mainly embraces the case $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{\frac{7+\alpha}{3}}} = +\infty$, which is under the L^2-supercritical case, under this case $I(u)$ is no more bounded below on $\mathcal{S}_c$. Hence, we try to establish the existence of a critical point of I on $\mathcal{S}_c$ by considering minimization problem which is similar to the idea explored by [26],

$$m(c) := \inf_{u\in\mathcal{M}_c} I(u).$$

Remark 1.2. *When $K(x)$ is a constant, Ye [26] showed the monotonicity of $c \to m(c)$, using which Ye [26] could exclude the vanishing case and the dichotomy case of minimizing sequence. However, this method relies heavily on the power-type nonlinearity $f(u) = |u|^{p-2}u$ with $p \in (\frac{14}{3}, 6)$, when K is non-constant and with the appearance of convolution term, the method in [26] does not work any more. In this point of view, the paper extends and promotes the previous results on prescribed normlized solutions of the Kirchhoff-type equation with Hartree-type nonlinearity.*

To complete the Section 1, hereafter we briefly sketch our proof. Note that we could obtain the following remarks from (K2) and (F3).

Remark 1.3. *For any $x \in \mathbb{R}^3$ and $t > 0$, from* (K2)*, we can easily deduce that*

$$L_0(x,t) := t^{\frac{3\theta-3-\alpha}{2}}[K(t^{-1}x) - K(x)] - \frac{2(1 - t^{\frac{3\theta-3-\alpha}{2}})}{3\theta - 3 - \alpha}\nabla K(x) \cdot x \geq 0.$$

By the continuity of $L_0(x, \cdot)$, we have

$$\lim_{t\to 0} L_0(x,t) = -\frac{2}{3\theta - 3 - \alpha}\nabla K(x) \cdot x \geq 0, \quad \forall\ x \in \mathbb{R}^3,$$

which implies for any $x \in \mathbb{R}^3$, $t \mapsto K(tx)$ is nonincreasing on $(0, \infty)$. Since $L_0(x, 2) \geq 0$ for all $x \in \mathbb{R}^3$, we have

$$0 \leq -\nabla K(x) \cdot x \leq \frac{2^{\frac{3\theta-5-\alpha}{2}}(3\theta - 3 - \alpha)[K(x/2) - K(x)]}{2^{\frac{3\theta-3-\alpha}{2}} - 1}, \quad \forall\ x \in \mathbb{R}^3.$$

Thus, by letting $|x| \to \infty$ we can conclude that $|\nabla K(x) \cdot x| \to 0$.

Remark 1.4. *From* (F3)*, for any $t > 0$ and $\tau \in \mathbb{R}$,*

$$L_1(t,\tau) := \frac{2(1 - t^{\frac{7+\alpha-3\theta}{2}})}{7 + \alpha - 3\theta}[f(\tau)\tau - \theta F(\tau)] - \frac{2\alpha + 14 - 6\theta}{3(7 + \alpha - 3\theta)}F(\tau) + \frac{2}{3}t^{-\frac{3\theta}{2}}F(t^{\frac{3}{2}}\tau)$$

is non-negative. In particular, for any $\tau \in \mathbb{R} \setminus \{0\}$, by (F3)*, we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L_1(t,\tau) &= t^{\frac{5+\alpha-3\theta}{2}}|\tau|^{\frac{7+\alpha}{2}}\left[\frac{f(t^{\frac{3}{2}}\tau)t^{\frac{3}{2}}\tau - \theta F(t^{\frac{3}{2}}\tau)}{|t^{\frac{3}{2}}\tau|^{\frac{7+\alpha}{3}}} - \frac{f(\tau)\tau - \theta F(\tau)}{|\tau|^{\frac{7+\alpha}{3}}}\right]\\ &\begin{cases}\geq 0, & t \geq 1;\\ \leq 0, & 0 < t < 1,\end{cases}\end{aligned}$$

which implies that $L_1(t,\tau) \geq L_1(1,\tau) = 0$ for all $t > 0$ and $\tau \in \mathbb{R}$. By the continuity of $L_1(\cdot,\tau)$, we obtain

$$L_1(0,\tau) := \lim_{t\to 0^+} h(t,\tau) = \frac{2}{5+\alpha-3\theta}\left[f(\tau)\tau - \frac{7+\alpha}{3}F(\tau)\right] \geq 0, \quad \forall\, \tau \in \mathbb{R}$$

together with

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{F(t)}{|t|^{\frac{4+\alpha}{3}}t} = \frac{1}{|t|^{\frac{10+\alpha}{3}}}\left[f(t)t - \frac{7+\alpha}{3}F(t)\right]$$

imply that $\frac{F(t)}{|t|^{\frac{4+\alpha}{3}}t}$ is nondecreasing on both $(-\infty,0)$ and $(0,+\infty)$.

Under the help of Remarks 1.3 and 1.4, then we could establish a new inequality in Lemma 3.7, with which we could prove that $m(c)$ is nonincreasing and $m(c) > m(\tilde{c})$ for any $\tilde{c} > c$ provided $m(c)$ is attained. Besides, due to the appearance of potential $K(x)$, we need to introduce the following limit equation,

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u - \lambda u = \left(\displaystyle\int_{\mathbb{R}^3}\frac{K_\infty F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}y\right)K_\infty f(u(x)), \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

and the corresponding energy functional is

$$\begin{aligned} I^\infty(u) =& \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2 \\ & - \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K_\infty^2 F(u(y))F(u(x))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Comparing the $m(c)$ with $m^\infty(c)$, where

$$m^\infty(c) = \inf_{u\in\mathcal{M}_c^\infty} I^\infty(u).$$

and

$$\mathcal{M}_c^\infty := \left\{u \in \mathcal{S}_c : J^\infty(u) := \frac{d}{dt}I^\infty(u^t)\big|_{t=1} = 0\right\},$$

we could overcome the difficulty caused by the lack of compactness of Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ for $2 \leq s < 6$, and we can verify that $m(c)$ is achieved by $\bar{u}_c$. Then by a standard method we could prove the $\bar{u}_c$ is the critical point of $I(u)$ on $\mathcal{S}_c$.

The organization of the remainder of this paper is as follows. In Sect. 2, we are devoted to establishing the essential inequality by Lemma 3.7 and studying the characteristic description of $m(c)$ in Lemma 3.10. In Sect. 3, we introduce several energy comparisons which are vital to prove the Theorem 1.1, and show the u satisfying $I(u) = m(c)$ is the critical point of $I|_{\mathcal{S}_c}$.

Throughout this paper we make use of the following notations:

• $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall\ u, v \in H^1(\mathbb{R}^3);$$

• $L^s(\mathbb{R}^3)(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^3} |u|^s \mathrm{d}x)^{\frac{1}{s}}$;

• For any $u \in H^1(\mathbb{R}^3)$, $u^t(x) := t^{3/2}u(tx)$ and $u_t(x) := t^{1/2}u(x/t)$;

• For any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$;

• $\mathcal{C}_1, \mathcal{C}_2, \cdots$ denote positive constants possibly different in different places.

2 Preliminary results

Noting that by the scaling, we have

$$\begin{aligned} I(u^t) =& \frac{at^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{bt^4}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \\ & - \frac{1}{2} t^{-3-\alpha} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(t^{-1}x)K(t^{-1}y)F(t^{\frac{3}{2}}u(x))F(t^{\frac{3}{2}}u(y))}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \end{aligned}$$

and the corresponding limit form is

$$\begin{aligned} I^\infty(u^t) =& \frac{at^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{bt^4}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \\ & - \frac{1}{2} t^{-3-\alpha} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K_\infty^2 F(t^{\frac{3}{2}}u(x))F(t^{\frac{3}{2}}u(y))}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Recalling $J(u) = \frac{\mathrm{d}}{\mathrm{d}t} I(u^t)\big|_{t=1}$ and $J^\infty(u) := \frac{\mathrm{d}}{\mathrm{d}t} I^\infty(u^t)\big|_{t=1}$, then we have

$$\begin{aligned} J(u) =& a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 \\ & - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x)K(y)F(u(y))}{|x-y|^{3-\alpha}} \left[\frac{3}{2} f(u(x))u(x) - \frac{3+\alpha}{2} F(u(x)) \right] \mathrm{d}x\mathrm{d}y \\ & + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla K(x) \cdot xK(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \end{aligned}$$

and the corresponding limit form is

$$
\begin{aligned}
J^{\infty}(u) =& a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 \\
& - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_\infty^2 F(u(y))}{|x-y|^{3-\alpha}}\left[\frac{3}{2}f(u(x))u(x) - \frac{3+\alpha}{2}F(u(x))\right]\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

3 The proof of Theorem 1.1

To complete the proof, we first show some energy comparison.

Lemma 3.1. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then $m(c)$ is non-increasing on $(0,\infty)$. In particular, if $m(c)$ is achieved, then $m(c) > m(\tilde{c})$ for any $\tilde{c} > c$.*

Proof. For any $c_2 > c_1 > 0$, there exists $\{u_n\} \subset \mathcal{M}_{c_1}$ such that

$$
I(u_n) < m(c_1) + \frac{1}{n}.
$$

Let $\xi = \sqrt{c_2/c_1} \in (1,\infty)$ and $v_n(x) = \xi^{-1/2}u_n(\xi^{-1}x)$. Then $\|v_n\|_2^2 = c_2$ and $\|\nabla v_n\|_2 = \|\nabla u_n\|_2$. By Lemma 3.10, there exists $t_n > 0$ such that $(v_n)^{t_n} \in \mathcal{M}_{c_2}$. Note that since $(3+\alpha-\mu)K(x) + 2\nabla K(x)\cdot x \geq 0$ for all $x \in \mathbb{R}^3$ then

$$
t \mapsto t^{\frac{3+\alpha-\mu}{2}}K(tx) \textit{ is nondecreasing on } (0,+\infty) \textit{ for every } x \in \mathbb{R}^3. \tag{3.1}
$$

Next, by (K2),(3.8) and (3.1), it follows that

$$
\begin{aligned}
& m(c_2) \\
\leq & I((v_n)^{t_n}) \\
= & \frac{at_n^2}{2}\|\nabla u_n\|_2^2 + \frac{bt_n^4}{2}\|\nabla u_n\|_2^4 \\
& - \frac{1}{2}\xi^{3+\alpha}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(t_n^{-1}\xi x)K(t_n^{-1}\xi y)F(t_n^{\frac{3}{2}}\xi^{-\frac{1}{2}}u_n(x))F(t_n^{\frac{3}{2}}\xi^{-\frac{1}{2}}u_n(y))}{t_n^{3+\alpha}|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
= & I((u_n)^{t_n}) + \frac{1}{2}t_n^{-(3+\alpha)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(t_n^{-1}x)K(t_n^{-1}y)F(t_n^{\frac{3}{2}}u_n(x))F(t_n^{\frac{3}{2}}u_n(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& - \frac{1}{2}\int_{\mathbb{R}^6}\frac{\xi^{3+\alpha-\mu}K(t_n^{-1}\xi x)K(t_n^{-1}\xi y)\xi^{\mu}F(t_n^{\frac{3}{2}}\xi^{-\frac{1}{2}}u_n(x))F(t_n^{\frac{3}{2}}\xi^{-\frac{1}{2}}u_n(x))}{t_n^{3+\alpha}|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
\leq & I(u_n) - \frac{a(1-t_n^2)^2}{4}\|\nabla u_n\|_2^2 < m(c_1) + \frac{1}{n},
\end{aligned}
$$

which implies $m(c_2) \leq m(c_1)$ by letting $n \to \infty$.

We now assume that $m(c)$ is achieved, that is, there exists $u \in \mathcal{M}_c$ such that $I(u) = m_c$ for any given $c < \tilde{c}$. Let $\tilde{\xi} = \tilde{c}/c \in (1,\infty)$ and $v(x) = \tilde{\xi}^{-1/2}u(\tilde{\xi}^{-1}x)$. Then $\|v\|_2^2 = \tilde{c}$ and $\|\nabla v\|_2 = \|\nabla u\|_2$. By lemma 3.10, there exists $\tilde{t} > 0$ such that $v^{\tilde{t}} \in \mathcal{M}_{\tilde{c}}$. Then it follows from (K2), (3.8) and (3.1) that

$$\begin{aligned}
& m(\tilde{c}) \\
\le\ & I(v^{\tilde{t}}) \\
=\ & \frac{a\tilde{t}^2}{2}\|\nabla u\|_2^2 + \frac{b\tilde{t}^4}{2}\|\nabla u\|_2^4 \\
& - \frac{1}{2}\tilde{\xi}^{3+\alpha}\tilde{t}^{-(3+\alpha)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(\tilde{t}^{-1}\tilde{\xi}x)K(\tilde{t}^{-1}\tilde{\xi}y)F(\tilde{t}^{\frac{3}{2}}\tilde{\xi}^{-\frac{1}{2}}u)F(\tilde{t}^{\frac{3}{2}}\tilde{\xi}^{-\frac{1}{2}}u)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
=\ & I((u)^{\tilde{t}}) + \frac{1}{2}\tilde{t}^{-(3+\alpha)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(\tilde{t}^{-1}x)K(\tilde{t}^{-1}y)F(\tilde{t}^{\frac{3}{2}}u)F(\tilde{t}^{\frac{3}{2}}u)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& - \frac{1}{2}\tilde{t}^{-(3+\alpha)}\int_{\mathbb{R}^6}\frac{\tilde{\xi}^{3+\alpha-\mu}K(\tilde{t}^{-1}\tilde{\xi}x)K(\tilde{t}^{-1}\tilde{\xi}y)\tilde{\xi}^{\mu}F(\tilde{t}^{\frac{3}{2}}\tilde{\xi}^{-\frac{1}{2}}u)F(\tilde{t}^{\frac{3}{2}}\tilde{\xi}^{-\frac{1}{2}}u)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
\le\ & I(u) - \frac{a(1-\tilde{t}^2)^2}{4}\|\nabla u\|_2^2 < m(c).
\end{aligned}$$

We have completed the proof. □

Corollary 3.2. *Assume that* (F1)-(F3) *hold. Then* $m^\infty(c)$ *is nonincreasing on* $(0,\infty)$. *In particular, if* $m^\infty(c)$ *is achieved, then* $m^\infty(c) > m^\infty(\tilde{c})$ *for any* $\tilde{c} > c$.

Lemma 3.3. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then* $m(c) \le m^\infty(c)$.

Proof. In view of Lemmas 3.10 and 3.13, we have $\mathcal{M}_c^\infty \neq \emptyset$ and $m^\infty(c) > 0$. Inspired by [7, 22], assume by contradiction that $m(c) > m^\infty(c)$. Let $\varepsilon := m(c) - m^\infty(c)$. Then there exists u_ε^∞ such that

$$u_\varepsilon^\infty \in \mathcal{M}_c^\infty \text{ and } m^\infty(c) + \frac{\varepsilon}{2} > I^\infty(u_\varepsilon^\infty).$$

In view of Lemma 3.10, there exists $t_\varepsilon > 0$ such that $(u_\varepsilon^\infty)^{t_\varepsilon} \in \mathcal{M}_c$. Since $K_\infty \le K(x)$ for all $x \in \mathbb{R}^3$, it follows from Corollary 3.8 that

$$m^\infty(c) + \frac{\varepsilon}{2} > I^\infty(u_\varepsilon^\infty) \ge I^\infty((u_\varepsilon^\infty)^{t_\varepsilon}) \ge I((u_\varepsilon^\infty)^{t_\varepsilon}) \ge m(c).$$

This contradiction shows that $m(c) \le m^\infty(c)$. □

Lemma 3.4. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then* $m(c)$ *is achieved.*

Proof. According to Lemmas 3.10 and 3.13, we have $\mathcal{M}_c \neq \emptyset$ and $m(c) > 0$. Let $\{u_n\} \subset \mathcal{M}_c$ be such that $I(u_n) \to m_c$. Since $J(u_n) = 0$, then it follows (3.8) from with $t \to 0$ that

$$m(c) + o(1) = I(u_n) \geq \frac{a}{4}\|\nabla u_n\|_2^2,$$

together with $\{u_n\} \subset \mathcal{S}_c$ imply that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^3$. There are two possible cases: i) $\bar{u} = 0$ and ii) $\bar{u} \neq 0$.
Case i) $\bar{u} = 0$, namely $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n \to 0$ a.e. in $\mathbb{R}^3$. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{[K(x)K(y) - K_\infty^2]F(u_n(x))F(u_n(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
=&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_\infty[K(x) - K_\infty]F(u_n(x))F(u_n(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)[K(y) - K_\infty]F(u_n(x))F(u_n(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

and by Hardy-Littlewood-Sobolev inequality, it is easily to check that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{[K(x) - K_\infty]K_\infty F(u_n(x))F(u_n(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\leq \mathcal{C}K_\infty \left(\int_{\mathbb{R}^3} \left|[K(x) - K_\infty]F(u_n)\right|^{\frac{6}{3+\alpha}}\mathrm{d}x\right)^{\frac{3+\alpha}{6}} \cdot \left(\int_{\mathbb{R}^3} |F(u_n)|^{\frac{6}{3+\alpha}}\mathrm{d}y\right)^{\frac{3+\alpha}{6}} \\
&\leq \tilde{\mathcal{C}}\left(\int_{\mathbb{R}^3} |K(x) - K_\infty|^{\frac{6}{3+\alpha}}|F(u_n)|^{\frac{6}{3+\alpha}}\mathrm{d}x\right)^{\frac{3+\alpha}{6}} \to 0,\ n \to \infty.
\end{aligned}$$

Recalling (K1) and Remark 1.3, we can obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{[K(x)K(y) - K_\infty^2]F(u_n(x))F(u_n(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
=&\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla K(x)\cdot xK(y)F(u_n(x))F(u_n(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y = 0,
\end{aligned}$$

from which it follows that

$$I^\infty(u_n) \to m(c), \quad J^\infty(u_n) \to 0. \tag{3.2}$$

From (3.2), Lemma 3.13-(i), one has

$$\begin{aligned}
a\rho_0^2 &\leq a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_\infty^2 F(u(y))}{|x-y|^{3-\alpha}}\left[\frac{3}{2}f(u(x))u(x) - \frac{3+\alpha}{2}F(u(x))\right]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Together with (3.2) and Lion's concentration-compactness principle [23, Lemma 1.21], we prove that there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 \mathrm{d}x > \delta$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$J^\infty(\hat{u}_n) = o(1), \quad I^\infty(\hat{u}_n) \to m(c), \quad \int_{B_1(0)} |\hat{u}_n|^2 \mathrm{d}x > \delta. \tag{3.3}$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, up to a subsequence, $\hat{u}_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^3)$, $\hat{u}_n \to \hat{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $s \in [1, 6)$ and $\hat{u}_n \to \hat{u}$ a.e. on $\mathbb{R}^3$. Let $w_n = \hat{u}_n - \hat{u}$, thus we have

$$\|\hat{u}\|_2^2 := \hat{c} \le c, \quad \|w_n\|_2^2 := \hat{c}_n \le c \text{ for large } n \in \mathbb{N},$$

and recalling [21, Lemma 2.10], [22, Lemma 2.7] and [23], we have the following Brezis-Lieb type lemma

$$I^\infty(\hat{u}_n) = I^\infty(\hat{u}) + I^\infty(w_n) + \frac{b}{2}\|\nabla \hat{u}\|_2^2 \|\nabla w_n\|_2^2 + o_n(1)$$

and

$$J^\infty(\hat{u}_n) = J^\infty(\hat{u}) + J^\infty(w_n) + b\|\nabla \hat{u}\|_2^2 \|\nabla w_n\|_2^2 + o_n(1).$$

Denote

$$\begin{aligned}
\Psi^\infty(u) &:= I^\infty(u) - \frac{1}{4} J^\infty(u) \\
&= \frac{a}{4}\|\nabla u\|_2^2 \\
&\quad + \frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} K_\infty^2 F(u(y))[3f(u(x))u(x) - (7+\alpha)F(u(x))]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By Remark 1.4, we have $\Psi^\infty(u) > 0$ for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. Moreover, it follows from (3.3) that

$$\Psi^\infty(w_n) \le m(c) - \Psi^\infty(\hat{u}) + o(1), \quad J^\infty(w_n) \le -J^\infty(\hat{u}) + o(1). \tag{3.4}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then it follows from Lemmas 3.1 and 3.3 that

$$m^\infty(\hat{c}) \le I^\infty(\hat{u}) = m(c) \le m(\hat{c}) \le m^\infty(\hat{c}), \quad J^\infty(\hat{u}) = 0,$$

which together with $m^\infty(c) \le m^\infty(\hat{c}) \le I^\infty(\hat{u}) = m(c) \le m^\infty(c)$, imply

$$I^\infty(\hat{u}) = m^\infty(\hat{c}) = m(\hat{c}) = m(c) = m^\infty(c), \quad J^\infty(\hat{u}) = 0.$$

On the other hand, we consider the case that $w_n \neq 0$, in view of Corollary 3.11, there exists $t_n > 0$ such that $(w_n)^{t_n} \in \mathcal{M}^{\infty}_{\hat{c}_n}$. From (3.4), Corollary 3.8, Lemmas 3.1 and 3.3 we can obtain that

$$\begin{aligned} m(c) - \Psi^{\infty}(\hat{u}) + o(1) \geq \Psi^{\infty}(w_n) &= I^{\infty}(w_n) - \frac{1}{4}J^{\infty}(w_n) \\ &\geq I^{\infty}((w_n)^{t_n}) - \frac{t_n^4}{4}J^{\infty}(w_n) \\ &\geq m^{\infty}(\hat{c}_n) - \frac{t_n^4}{4}J^{\infty}(w_n) \\ &\geq m^{\infty}(c) - \frac{t_n^4}{4}J^{\infty}(w_n) + o(1) \\ &\geq m(c) - \frac{t_n^4}{4}J^{\infty}(w_n) + o(1), \end{aligned}$$

which implies that $J^{\infty}(w_n) \geq 0$, otherwise we can get a contradiction by $\Psi^{\infty}(\hat{u}) > 0$. In view of (3.4), $J^{\infty}(\hat{u}) \leq 0$. In view of Corollary 3.11, there exists $t_{\infty} > 0$ such that $\hat{u}^{t_{\infty}} \in \mathcal{M}^{\infty}_{\hat{c}}$. From (3.8),(3.3), the weak semicontinuity of the norm, Fatou's lemma, Lemmas 3.1 and 3.3, one has

$$\begin{aligned} m(c) &= \lim_{n\to\infty} \left[I^{\infty}(\hat{u}_n) - \frac{1}{4}J^{\infty}(\hat{u}_n) \right] \\ &= \lim_{n\to\infty} \Psi^{\infty}(\hat{u}_n) \geq \Psi^{\infty}(\hat{u}) \\ &= I^{\infty}(\hat{u}) - \frac{1}{4}J^{\infty}(\hat{u}) \geq I^{\infty}(\hat{u}^{t_{\infty}}) - \frac{t_{\infty}^4}{4}J^{\infty}(\hat{u}) \\ &\geq m^{\infty}(\hat{c}) - \frac{t_{\infty}^4}{4}J^{\infty}(\hat{u}) \geq m(\hat{c}) \geq m(c), \end{aligned}$$

hence by Corollary 3.2, we know that

$$m^{\infty}(c) \leq m^{\infty}(\hat{c}) = I^{\infty}(\hat{u}) = m(\hat{c}) = m(c) \leq m^{\infty}(c), \quad J^{\infty}(\hat{u}) = 0.$$

Thus, $m^{\infty}(\hat{c})$ is achieved at $\hat{u}$. In view of Lemma 3.1, we deduce that $\|\hat{u}\|_2^2 = \hat{c} = c$ due to $m^{\infty}(\hat{c}) = m^{\infty}(c)$. By Lemma 3.10, there exists $\hat{t} > 0$ such that $\hat{u}^{\hat{t}} \in \mathcal{M}_c$. Then it follows from Corollary 3.8 that

$$\begin{aligned} m(c) &\leq I(\hat{u}^{\hat{t}}) \leq I^{\infty}(\hat{u}^{\hat{t}}) \\ &\leq I^{\infty}(\hat{u}) - \frac{a(1-\hat{t}^2)^2}{4}\|\nabla \hat{u}\|_2^2 = m(c) - \frac{a(1-\hat{t}^2)^2}{4}\|\nabla \hat{u}\|_2^2, \end{aligned}$$

which implies that $\hat{u} \in \mathcal{M}_c$ and $I(\hat{u}) = m(c)$. Hence, $m(c)$ is achieved at $\hat{u} \in \mathcal{M}_c$.

Case ii) $\bar{u} \neq 0$. Let $v_n = u_n - \bar{u}$, hence

$$\|\bar{u}\|_2^2 := \bar{c} \leq c, \quad \|v_n\|_2^2 := c_n \leq c \text{ for large } n \in \mathbb{N}.$$

Then recalling [21, Lemma 2.10], [22, Lemma 2.7] and [23] again, we have

$$I(u_n) = I(\bar{u}) + I(v_n) + \frac{b}{2}\|\nabla \bar{u}\|_2^2 \|\nabla v_n\|_2^2 + o_n(1)$$

and

$$J(u_n) = J(\bar{u}) + J(v_n) + b\|\nabla \bar{u}\|_2^2 \|\nabla v_n\|_2^2 + o_n(1).$$

Denote

$$\begin{aligned}
\Psi(u) &:= I(u) - \frac{1}{4}J(u) \\
&= \frac{a}{4}\|\nabla u\|_2^2 \\
&\quad + \frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)F(u(y))}{|x-y|^{3-\alpha}}\left[3f(u(x))u(x) - (7+\alpha)F(u(x))\right] \mathrm{d}x\mathrm{d}y \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla K(x)\cdot xF(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Recalling that Remarks 1.3 and 1.4, thus $\Psi(u) > 0$ for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. In a similar way to the inequality (3.4),

$$\Psi(v_n) \leq m(c) - \Psi(\bar{u}) + o(1), \quad J(v_n) \leq -J(\bar{u}) + o(1). \tag{3.5}$$

If there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} = 0$, then it follows from Lemma 3.12 that

$$m(\bar{c}) \leq I(\bar{u}) = m(c) \leq m(\bar{c}), \quad J(\bar{u}) = 0,$$

which implies

$$I(\bar{u}) = m(c) = m(\bar{c}), \quad J(\bar{u}) = 0. \tag{3.6}$$

Otherwise, we consider the case that $v_n \neq 0$. In view of Lemma 3.10, there exists $t_n > 0$ such that $(v_n)^{t_n} \in \mathcal{M}_{c_n}$. From (3.8) and (3.5), we have

$$\begin{aligned}
m(c) - \Psi(\bar{u}) + o(1) \geq \Psi(v_n) &= I(v_n) - \frac{1}{4}J(v_n) \\
&\geq I((v_n)^{t_n}) - \frac{t_n^4}{4}J(v_n) \geq m(\bar{c}_n) - \frac{t_n^4}{4}J(v_n) \\
&\geq m(c) - \frac{t_n^4}{4}J(v_n) + o_n(1),
\end{aligned}$$

which implies that $J(v_n) \geq 0$, otherwise we can get a contradiction by $\Psi(\bar{u}) > 0$. In view of (3.5), $J(\bar{u}) \leq 0$. In view of Lemma 3.10, there exists $\tilde{t} > 0$ such that $\bar{u}^{\tilde{t}} \in \mathcal{M}_{\bar{c}}$. Then it follows from (3.8), the weak semicontinuity of norm, Fatou's lemma and Lemma 3.1 that

$$\begin{aligned} m(c) &= \lim_{n\to\infty} \left[I(u_n) - \frac{1}{4}J(u_n) \right] = \lim_{n\to\infty} \Psi(u_n) \\ &\geq \Psi(\bar{u}) = I(\bar{u}) - \frac{1}{4}J(\bar{u}) \\ &\geq I(\bar{u}^{\tilde{t}}) - \frac{\tilde{t}^4}{4}J(\bar{u}) \geq m(\bar{c}) \geq m(c), \end{aligned}$$

which implies (3.6) holds for $v_n \neq 0$. This shows that $m(\bar{c})$ is achieved at $\bar{u} \in \mathcal{M}_{\bar{c}}$. In view of Lemma 3.1, we have $\|\bar{u}\|_2^2 = \bar{c} = c$ due to $m(c) = m(\bar{c})$. By Lemma 3.10, there exists $\bar{t} > 0$ such that $\bar{u}^{\bar{t}} \in \mathcal{M}_c$. Then it follows from (3.8), (3.1) and (3.6) that

$$\begin{aligned} m(c) &\leq I(\bar{u}^{\bar{t}}) \\ &\leq I(\bar{u}) - \frac{a(1-\bar{t}^2)^2}{4}\|\nabla\bar{u}\|_2^2 = m(c) - \frac{a(1-\bar{t}^2)^2}{4}\|\nabla\bar{u}\|_2^2, \end{aligned}$$

which implies that $\bar{u} \in \mathcal{M}_c$ and $I(\bar{u}) = m(c)$. Hence, $m(c)$ is achieved at $\bar{u} \in \mathcal{M}_c$. □

In the same way as [7] or [21], we can obtain the following Lemma.

Lemma 3.5. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. If* $\bar{u} \in \mathcal{M}_c$ *and* $I(\bar{u}) = m(c)$*, then* $\bar{u}$ *is a critical point of* $I|_{\mathcal{S}_c}$.

Lemma 3.6. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. If* $\bar{u} \in \mathcal{S}_c$ *is a critical point of* $I|_{\mathcal{S}_c}$*, then* $J(\bar{u}) = 0$*, and there exists* $\lambda_c < 0$ *such that* $I'(\bar{u}) - \lambda_c\bar{u} = 0$.

Proof. Since $(I|_{\mathcal{S}_c})'(\bar{u}) = 0$, there exists $\lambda_c \in \mathbb{R}$ such that $I'(\bar{u}) - \lambda_c\bar{u} = 0$, and so

$$\begin{aligned} \langle I'(u) - \lambda_c\bar{u}, \bar{u}\rangle =& a\|\nabla\bar{u}\|_2^2 + b\|\nabla\bar{u}\|_2^4 \\ &- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)F(\bar{u}(y))K(x)f(\bar{u}(x))\bar{u}(x)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\ &- \lambda_c\|\bar{u}\|_2^2 = 0. \end{aligned} \tag{3.7}$$

Moreover, $\bar{u}$ satisfies the following Pohozaev identity:

$$\begin{aligned}\mathcal{P}(\bar{u}) :=& \frac{a}{2}\|\nabla\bar{u}\|_2^2 + \frac{b}{2}\|\nabla\bar{u}\|_2^4 \\ &- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)F(\bar{u}(y))F(\bar{u}(x))[(3+\alpha)K(x) + 2\nabla K(x)\cdot x]}{2|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\ &- \frac{3}{2}\lambda_c\|\bar{u}\|_2^2 = 0,\end{aligned}$$

together with (3.7), then we have

$$J(\bar{u}) = \frac{3}{2}\langle I'(u) - \lambda_c\bar{u}, \bar{u}\rangle - \mathcal{P}(\bar{u}) = 0.$$

Noting that $\|\bar{u}\|_2^2 = c$, it follows (F1) and (K2) that

$$\begin{aligned}&2\lambda_c c \\ =& \int_{\mathbb{R}^6} \frac{K(y)F(\bar{u}(y))[K(x)f(\bar{u}(x))\bar{u}(x) - (3+\alpha)K(x)F(\bar{u}(x))]}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\ &- \int_{\mathbb{R}^6} \frac{2K(y)F(\bar{u}(y))\nabla K(x)\cdot xF(\bar{u}(x))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\ =& \int_{\mathbb{R}^6} \frac{K(y)F(\bar{u}(y))[K(x)f(\bar{u}(x))\bar{u}(x) - \mu F(\bar{u}(x))]}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\ &- \int_{\mathbb{R}^6} \frac{K(y)F(\bar{u}(y))[(3+\alpha-\mu)K(x)F(\bar{u}(x)) + 2\nabla K(x)\cdot xF(\bar{u}(x))]}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\ <& \ 0,\end{aligned}$$

thus $\lambda_c < 0$. This completes the proof. $\square$

Proof of Theorem 1.1. According to the Lemmas 3.12 and 3.4-3.6, for any $c > 0$ there exists $\bar{u}_c \in \mathcal{M}_c$ such that

$$I(\bar{u}_c) = m(c) = \inf_{u\in\mathcal{S}_c}\max_{t>0} I((\bar{u}_c)^t) > 0, I'(\bar{u}_c) = 0,$$

and there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}^-$ such that $(\bar{u}_c, \lambda_c)$ is a solution of problem (1.1). $\square$

Lemma 3.7. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then*

$$I(u) \geq I(u^t) + \frac{1-t^4}{4}J(u) + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2, \quad \forall\ u \in H^1(\mathbb{R}^3),\ t > 0. \quad (3.8)$$

Proof. For any $x \in H^1(\mathbb{R}^3)$ and $t>0$, it is easily checked that

$$
\begin{aligned}
& I(u)-I(u^t) \\
= & \frac{a(1-t^2)}{2}\|\nabla u\|_2^2+\frac{b(1-t^4)}{2}\|\nabla u\|_2^4 \\
& -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& +\frac{t^{-3-\alpha}}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(t^{-1}x)K(t^{-1}y)F(t^{\frac{3}{3}}u(x))F(t^{\frac{3}{2}}u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
= & \frac{1-t^4}{4}\left\{a\|\nabla u\|_2^2+b\|\nabla u\|_2^4\right. \\
& \left.+\frac{3+\alpha}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y\right\} \\
& -\frac{1-t^4}{4}\cdot\frac{3}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)f(u(x))u(x)F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& +\frac{1-t^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nabla K(x)\cdot xK(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y+\frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 \\
& +\frac{t^{-(3+\alpha)}}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(t^{-1}x)K(t^{-1}y)F(t^{\frac{3}{2}}u(x))F(t^{\frac{3}{2}}u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& -\left[\frac{1}{2}+\frac{(3+\alpha)}{2}\cdot\frac{1-t^4}{4}\right]\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& -\frac{1-t^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nabla K(x)\cdot xK(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& +\frac{3}{2}\cdot\frac{1-t^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)f(u(x))u(x)F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

It suffices to prove that for any $x,y\in\mathbb{R}^3$, $t>0$ and $\tau_1,\tau_2:\mathbb{R}^3\to\mathbb{R}$,

$$
\begin{aligned}
& L_2(x,y,t,\tau_1,\tau_2) \\
:= & \frac{t^{-(3+\alpha)}}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(t^{-1}x)K(t^{-1}y)F(t^{\frac{3}{2}}\tau_1)F(t^{\frac{3}{2}}\tau_2)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& -\left[\frac{1}{2}+\frac{(3+\alpha)}{2}\cdot\frac{1-t^4}{4}\right]\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F(\tau_1)F(\tau_2)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& -\frac{1-t^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\nabla K(x)\cdot xK(y)F(\tau_1)F(\tau_2)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
& +\frac{3}{2}\cdot\frac{1-t^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)f(\tau_1)\tau_1F(\tau_2)}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y\geq 0.
\end{aligned}
$$

Let $M = t^3|\tau_1|^{\frac{\alpha+7}{3}}|\tau_2|^{\frac{\alpha+7}{3}}$, then we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}L_2(x,y,t,\tau_1,\tau_2)\\
=&t^{-4-\alpha}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(t^{-1}x)K(t^{-1}y)F(t^{\frac{3}{2}}\tau_2)\left[\frac{3}{2}f(t^{\frac{3}{2}}\tau_1)t^{\frac{3}{2}}\tau_1-\frac{3\theta}{2}F(t^{\frac{3}{2}}\tau_1)\right]\mathrm{d}x\mathrm{d}y\\
&+t^{-4-\alpha}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(t^{-1}y)F(t^{\frac{3}{2}}\tau_1)F(t^{\frac{3}{2}}\tau_2)\left[\left(\frac{3\theta}{2}-\frac{3+\alpha}{2}\right)K(t^{-1}x)\right.\\
&\left.-\nabla K(t^{-1}x)\cdot t^{-1}x\right]\mathrm{d}x\mathrm{d}y\\
&-t^3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(x)K(y)F(\tau_1)F(\tau_2)\left[\frac{3}{2}f(\tau_1)\tau_1-\frac{3\theta}{2}F(\tau_1)\right]\mathrm{d}x\mathrm{d}y\\
&-t^3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(y)F(\tau_1)F(\tau_2)\left[\left(\frac{3\theta}{2}-\frac{3+\alpha}{2}\right)K(x)-\nabla K(x)\cdot x\right]\mathrm{d}x\mathrm{d}y\\
=&\frac{3M}{2}\cdot\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(t^{-1}y)K(t^{-1}x)\left[\frac{f(t^{\frac{3}{2}}\tau_1)t^{\frac{3}{2}}\tau_1-\theta F(t^{\frac{3}{2}}\tau_1)}{|t^{\frac{3}{2}}\tau_1|^{\frac{\alpha+7}{3}}}\cdot\frac{F(t^{\frac{3}{2}}\tau_2)}{|t^{\frac{3}{2}}\tau_2|^{\frac{\alpha+7}{3}}}\right]\mathrm{d}x\mathrm{d}y\\
&+M\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(t^{-1}y)\frac{F(t^{\frac{3}{2}}\tau_1)}{|t^{\frac{3}{2}}\tau_1|^{\frac{\alpha+7}{3}}}\frac{F(t^{\frac{3}{2}}\tau_2)}{|t^{\frac{3}{2}}\tau_2|^{\frac{\alpha+7}{3}}}\left[\left(\frac{3\theta-3-\alpha}{2}\right)K(t^{-1}x)\right.\\
&\left.-\nabla K(t^{-1}x)\cdot t^{-1}x\right]\mathrm{d}x\mathrm{d}y\\
&-\frac{3M}{2}\cdot\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(y)K(x)\left[\frac{f(\tau_1)\tau_1-\theta F(\tau_1)}{|\tau_1|^{\frac{\alpha+7}{3}}}\cdot\frac{F(\tau_2)}{|\tau_2|^{\frac{\alpha+7}{3}}}\right]\mathrm{d}x\mathrm{d}y\\
&-M\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(y)\frac{F(\tau_1)}{|\tau_1|^{\frac{\alpha+7}{3}}}\frac{F(\tau_2)}{|\tau_2|^{\frac{\alpha+7}{3}}}\left[\left(\frac{3\theta-3-\alpha}{2}\right)K(x)-\nabla K(x)\cdot x\right]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Recalling the (F3), (K2), Remarks 1.3 and 1.4, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}L_2(x,y,t,\tau_1,\tau_2)\begin{cases}\geq 0, & t\geq 1;\\ \leq 0, & 0<t<1.\end{cases}$$

It follows that $L_2(x,y,t,\tau_1,\tau_2)\geq L_2(x,y,1,\tau_1,\tau_2)=0$ for all $x,y\in\mathbb{R}^3$, $t>0$ and $\tau_1,\tau_2\in\mathbb{R}$. We have completed this lemma. $\square$

From Lemma 3.7, we have the following corollaries.

Corollary 3.8. *Assume that* (F1)-(F3) *hold. Then*

$$I^\infty(u)\geq I^\infty(u^t)+\frac{1-t^4}{4}J^\infty(u)+\frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2,\quad \forall\ u\in H^1(\mathbb{R}^3),\ t>0.$$

Corollary 3.9. *Assume that* (K1), (K2) *and* (F1)-(F3) *hold. Then*

$$I(u) = \max_{t>0} I(u^t), \quad \forall\, u \in \mathcal{M}_c.$$

Lemma 3.10. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then for any* $u \in H^1(\mathbb{R}^3) \setminus \{0\}$*, there exists a unique* $t_u > 0$ *such that* $u^{t_u} \in \mathcal{M}_c$*.*

Proof. Let $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ be fixed and define a function $\zeta(t) := I(u^t)$ on $(0, \infty)$. Clearly, we have

$$\begin{aligned}
\zeta'(t) = 0 \Leftrightarrow\ & at\|\nabla u\|_2^2 + bt^3\|\nabla u\|_2^4 \\
&+ \frac{3+\alpha}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(t^{-1}x)K(t^{-1}y)F(t^{\frac{3}{2}}u(x))F(t^{\frac{3}{2}}u(y))}{t^{4+\alpha}|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\nabla K(t^{-1}x)\cdot(t^{-1}x)K(t^{-1}y)F(t^{\frac{3}{2}}u(x))F(t^{\frac{3}{2}}u(y))}{t^{4+\alpha}|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
&- \frac{3}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(t^{-1}x)K(t^{-1}y)f(t^{\frac{3}{2}}u(x))t^{\frac{3}{2}}u(x)F(t^{\frac{3}{2}}u(y))}{t^{4+\alpha}|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
\Leftrightarrow\ & \frac{1}{t}J(u^t) = 0 \Leftrightarrow u^t \in \mathcal{M}_c.
\end{aligned}$$

Note that Remarks 1.3 and 1.4 lead to

$$K(t^{-1}x)F(t^{\frac{3}{2}}\tau) \le t^{\frac{7+\alpha}{2}}K(x)F(\tau), \quad \forall\, x \in \mathbb{R}^3,\ t \in (0,1),\ \tau \in \mathbb{R}.$$

Thus, for any $t \in (0,1)$, one has

$$I(u^t) \ge \frac{at^2}{2}\|\nabla u\|_2^2 + \frac{bt^4}{2}\|\nabla u\|_2^4 - \frac{1}{2}t^4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y,$$

which implies that $\zeta(t) > 0$ for $t > 0$ small. Moreover, by (K1)-(K2) and (F1)-(F2), it is easy to verify that $\lim_{t\to 0}\zeta(t) = 0$ and $\zeta(t) < 0$ for t large. Therefore $\max_{t\in(0,\infty)}\zeta(t)$ is achieved at $\zeta_u > 0$ so that $\zeta'(t_u) = 0$ and $u^{t_u} \in \mathcal{M}_c$.

Next, we claim that t_u is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. Otherwise, for any given $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exist positive constants $t_1 \neq t_2$ such that $u^{t_1}, u^{t_2} \in \mathcal{M}_c$, that is, $J(u^{t_1}) = J(u^{t_2}) = 0$. Then (3.8) implies

$$I(u^{t_1}) > I(u^{t_2}) + \frac{t_1^4 - t_2^4}{4t_1^4}J(u^{t_1}) = I(u^{t_2}) > I(u^{t_1}) + \frac{t_2^4 - t_2^4}{4t_2^4}J(u^{t_2}) = I(u^{t_1}).$$

This contradiction shows that $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. $\square$

Similarly, we have the following corollary.

Corollary 3.11. *Assume that* (F1)-(F3) *hold. Then for any* $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, *there exists a unique* $t_u^\infty > 0$ *such that* $u^{t_u^\infty} \in \mathcal{M}_c^\infty$.

Combining Corollary 3.9 and Lemma 3.10, we obtain the following property.

Lemma 3.12. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then*

$$\inf_{u\in\mathcal{M}_c} I(u) = m(c) = \inf_{u\in\mathcal{S}_c}\max_{t>0} I(u^t).$$

Lemma 3.13. *Assume that* (K1),(K2) *and* (F1)-(F3) *hold. Then*
(i) *there exists* $\rho_0 > 0$ *such that* $\|\nabla u\|_2 \geq \rho_0$, $\forall\ u \in \mathcal{M}_c$;
(ii) $m(c) = \inf_{u\in\mathcal{M}_c} I(u) > 0$.

Proof. **(i)** By (F1), we deduce that

$$\frac{F(t)}{|t|^{\mu-1}t} \text{ is nonincreasing on both } (-\infty, 0) \text{ and } (0, +\infty),$$

and by Remark 1.4, we derive that for any $s \in \mathbb{R}$,

$$\begin{cases} |t|^\mu F(s) \leq F(st) \leq |t|^{\frac{7+\alpha}{3}} F(s), & \text{if } |t| \leq 1; \\ |t|^{\frac{7+\alpha}{3}} F(s) \leq F(st) \leq |t|^\mu F(s), & \text{if } |t| \geq 1, \end{cases}$$

which implies that there exists a constant $\mathcal{C}_0 > 0$ such that

$$0 \leq F(t) \leq \mathcal{C}_0(|t|^{\frac{7+\alpha}{3}} + |t|^\mu),\ \forall\ t \in \mathbb{R}.$$

Since $J(u) = 0$, $\forall\ u \in \mathcal{M}_c$, from the Gagliardo-Nirenberg inequality and the Hardy-Littlewood-Sobolev inequality, we deduce that

$$\begin{aligned}
&a\|\nabla u\|_2^2 \\
&\leq a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)F(u(y))}{|x-y|^{3-\alpha}}\left(\frac{3}{2}f(u(x))u(x) - \frac{3+\alpha}{2}F(u(x))\right)\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\nabla K(x)\cdot x)K(y)F(u(x))F(u(y))}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\leq \mathcal{C}_1\left(\|u\|_{\frac{14+2\alpha}{3+\alpha}}^{\frac{14+2\alpha}{3}} + \|u\|_{\frac{6\mu}{3+\alpha}}^{2\mu}\right) \\
&\leq \mathcal{C}_2\|\nabla u\|_2^4\|u\|_2^{\frac{2+2\alpha}{3+\alpha}} + \mathcal{C}_3\|\nabla u\|_2^{6\mu(3\mu-\alpha-3)}\|u\|_2^{6\mu(3+\alpha-\mu)} \\
&= \mathcal{C}_2 c^{\frac{1+\alpha}{3+\alpha}}\|\nabla u\|_2^4 + \mathcal{C}_3 c^{3\mu(3+\alpha-\mu)}\|\nabla u\|_2^{6\mu(3\mu-\alpha-3)},\ \forall\ u \in \mathcal{M}_c,
\end{aligned}$$

which concludes the proof of (i).

(ii) By (i) and (3.8) with $t \to 0$, we have

$$I(u) = I(u) - \frac{1}{4}J(u) \geq \frac{a}{4}\|\nabla u\|_2^2 \geq \frac{a}{4}\rho_0^2, \ \forall \ u \in \mathcal{M}_c.$$

Hence, $m(c) = \inf_{u \in \mathcal{M}_c} I(u) > 0$. □

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